

Counting and Tensorial Properties of Twist-Two Helicity-Flip Nucleon Form Factors

Zhang Chen^{1,*} and Xiangdong Ji^{2,†}

¹*Department of Physics, Manhattanville College, Purchase, NY 10577, USA*

²*Department of Physics, University of Maryland,
College Park, Maryland 20742, USA*

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Abstract

We perform a systematic analysis on the off-forward matrix elements of the twist-two quark and gluon helicity-flip operators. By matching the allowed quantum numbers and their crossing channel counterparts (a method developed by Ji & Lebed), we systematically count the number of independent nucleon form factors in off-forward scattering of matrix elements of these quark and gluon spin-flip operators. In particular, we find that the numbers of independent nucleon form factors twist-2, helicity flip quark (gluon) operators are $2n - 1$ ($2n - 5$) if n is odd, and $2n - 2$ ($2n - 6$) if n is even, with $n \geq 2$ ($n \geq 4$). We also analysis and write down the tensorial/Lorentz structure and kinematic factors of the expansion of these operators' matrix elements in terms of the independent form factors. These generalized form factors define the off-forward quark and gluon helicity-flip distributions in the literature.

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*Electronic address: chenz@mville.edu

†Electronic address: xji@physics.umd.edu

I. INTRODUCTION

To fully describe a nucleon in quantum chromodynamics (QCD), one needs to know the matrix elements of all possible quark and gluon operators involving the nucleon state. Among these operators, the matrix elements of those of twist-two, often having clear physical interpretation (e.g., corresponding to the energy-momentum tensor), give the leading contribution (thus they are often referred to as leading-twist) in appropriate hard processes. They are also more accessible to experimental measurement and are relatively simple.

The matrix elements can be taken between states of equal momenta (forward) or unequal momenta (off-forward), and contain valuable dynamical information about the internal structure of the nucleon. In the forward case, after extracting out the tensorial/Lorentz structure and kinematic factors, one obtains the irreducible matrix elements. These are (combinations of) moments of, and thus can be used to define, the conventional Feynman parton distributions, (eg, see [1]) On the other hand, the off-forward matrix elements are expanded in terms of (generalized) nucleon form factors (see Eq. (8)), which are closely connected to the moments of, and can be used to define a new type of parton distributions—the (generalized) off-forward parton distributions. At the same time, from the point view of the low-energy nucleon structure, these off-forward distributions can be considered as the generating functions for the form factors of the twist-two operators. In recent years, these (generalized) off-forward parton distributions, or simply generalized parton distributions (GPDs), of hadrons, especially those of the nucleon, have been the subject of much theoretical and experimental effort [2, 3].

As characterizations of certain properties exhibited by the nucleon in classes of (most often non-forward) high-energy scattering, e.g., deeply virtual Compton scattering (DVCS) and diffractive electroproduction of vector mesons, GPDs represent the low-energy internal structure of the particle. They are generalizations of both the Feynman parton distributions and the elastic electromagnetic form factors. In general the GPDs have their physical interpretations in the Fourier space as the quantum phase-space parton distributions [4]. Most interestingly, the distributions contain information about the orbital motion of partons in a (polarized) nucleon. For instance, knowing certain off-forward matrix elements and extracting the related GPDs allows for deduction of the quark and gluon orbital and spin contributions to the nucleon spin [5].

Therefore, the study of the generalized nucleon form factors is of much interest and importance. One of the essential understandings lies with the enumeration of independent nucleon form factors of twist-two operators, as well as the Lorentz structure and kinematic factors of the off-forward matrix elements, namely, its expansion into form factors. In [6], a method was developed to systematically count the number of hadronic form factors based on the partial wave formalism and crossing symmetry. There the case for spin-independent operators was discussed. The class of twist-two operators which depends on parton helicity change are the subject of this paper. We will enumerate the number of independent form factors for (the off-forward matrix elements of) quark and gluon helicity-flip operators, and write down the form factor expansion for the general quark operators.

The outline of the paper is as follows. In Section II, we give a brief review of the definitions of the twist-two operators and their matrix elements, as well as their relationship to the GPDs. Section III contains the systematic enumeration of independent form factors of the quark operators, while Section IV presents the form factor expansion for the quark helicity-flip operators. Section V does the same for the gluon operators. We conclude the

paper by giving the summary and outlook in Section VI. In Appendix A we provide a general discussion on tensorial properties of these operators and their representations. And in Appendix B, using the quark operators as examples, we give a discussion on the possible constraints Hermiticity and time reversal invariance requirements might impose.

II. THE TWIST-TWO HELICITY-FLIP QUARK AND GLUON OPERATORS

Using the now quite standard notation (see, eg, [2], [7]), we parameterize the kinematics of the off-forward scattering process as follows. The momenta and spins of the initial and final nucleons are P, S and P', S' , respectively. The four-momentum transfer $\Delta^\mu = P'^\mu - P^\mu$ has both longitudinal and transverse components, and its invariant is $t = \Delta^2$. Define a special system of coordinates in which the average nucleon momentum $\bar{P}^\mu = (P' + P)^\mu/2$ is collinear and in the z direction. Further define, as usual, two light-like four-vectors p^μ and n^μ with $p^2 = n^2 = 0$ and $p \cdot n = 1$. We have

$$\begin{aligned}\bar{P}^\mu &= (P' + P)^\mu/2 = p^\mu + (\bar{M}^2/2)n^\mu, \\ \Delta^\mu &= P'^\mu - P^\mu = -2\xi(p^\mu - (\bar{M}^2/2)n^\mu) + \Delta_\perp^\mu, \\ \bar{M}^2 &= M^2 - \Delta^2/4.\end{aligned}\tag{1}$$

The initial nucleon and parton have longitudinal momentum fractions $1 + \xi$ and $x + \xi$, respectively.

The following tower of twist-two operators is a generalization of the electromagnetic current

$$\mathcal{O}_q^{\mu_1 \cdots \mu_n} = \bar{\psi}_q(0) i \overleftrightarrow{\mathcal{D}}^{(\mu_1} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} \gamma^{\mu_n)} \psi_q(0),\tag{2}$$

where all indices are symmetrized and traceless (indicated by (\cdots)) and $\overleftrightarrow{\mathcal{D}} = (\overrightarrow{\mathcal{D}} - \overleftarrow{\mathcal{D}})/2$, with \mathcal{D} as the covariant derivative in QCD. An expansion of the off-forward matrix elements of these operators give rise to nucleon form factors whose linear combinations (with powers of ξ as coefficients) are the moments of the GPDs $H(x, \xi, t)$ and $E(x, \xi, t)$. Similarly, there are five additional towers of twist-two operators in QCD besides that in Eq. (2):

$$\begin{aligned}\tilde{\mathcal{O}}_q^{\mu_1 \cdots \mu_n} &= \bar{\psi}_q i \overleftrightarrow{\mathcal{D}}^{(\mu_1} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} \gamma^{\mu_n)} \gamma_5 \psi_q, \\ \mathcal{O}_{qT}^{\mu_1 \cdots \mu_n \alpha} &= \bar{\psi}_q i \overleftrightarrow{\mathcal{D}}^{(\mu_1} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} \sigma^{\mu_n) \alpha} \psi_q, \\ \mathcal{O}_g^{\mu_1 \cdots \mu_n} &= F^{(\mu_1 \alpha} i \overleftrightarrow{\mathcal{D}}^{\mu_2} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} F_\alpha^{\mu_n)}, \\ \tilde{\mathcal{O}}_g^{\mu_1 \cdots \mu_n} &= F^{(\mu_1 \alpha} i \overleftrightarrow{\mathcal{D}}^{\mu_2} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} i \tilde{F}_\alpha^{\mu_n)}, \\ \mathcal{O}_{gT}^{\mu_1 \cdots \mu_n \alpha \beta} &= F^{(\mu_1 \alpha} i \overleftrightarrow{\mathcal{D}}^{\mu_2} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} F^{\mu_n) \beta}.\end{aligned}\tag{3}$$

The corresponding GPDs are labelled by $(\tilde{H}_q(x, \xi), \tilde{E}_q(x, \xi))$, $(H_{Tq}(x, \xi), E_{Tq}(x, \xi), \tilde{H}_{Tq}(x, \xi), \tilde{E}_{Tq}(x, \xi))$, $(H_g(x, \xi), E_g(x, \xi))$, $(\tilde{H}_g(x, \xi), \tilde{E}_g(x, \xi))$, and $(H_{Tg}(x, \xi), E_{Tg}(x, \xi), \tilde{H}_{Tg}(x, \xi), \tilde{E}_{Tg}(x, \xi))$, respectively [2, 3].

We concentrate our attention to the helicity-flip operators \mathcal{O}_{qT} and \mathcal{O}_{gT} and their corresponding form factors and GPDs. For example, it is known that for the lowest spin of each

kind, the GPDs arise from the following definition[3, 7, 8]

$$\begin{aligned}
\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P'S' | \bar{\psi}_q(-\frac{1}{2}\lambda n) \sigma^{\mu\nu} \psi_q(\frac{1}{2}\lambda n) | PS \rangle &= H_{Tq}(x, \xi) \bar{U}(P'S') \sigma^{\mu\nu} U(PS) \\
&+ \tilde{H}_{Tq}(x, \xi) \bar{U}(P'S') \frac{\bar{P}^{[\mu} i \Delta^{\nu]}}{M^2} U(PS) \\
&+ E_{Tq}(x, \xi) \bar{U}(P'S') \frac{\gamma^{[\mu} i \Delta^{\nu]}}{M} U(PS) \\
&+ \tilde{E}_{Tq}(x, \xi) \bar{U}(P'S') \frac{\gamma^{[\mu} i \bar{P}^{\nu]}}{M} U(PS) + \dots \quad (4)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{x} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P'S' | F^{(\mu\alpha}(-\frac{\lambda}{2}n) F^{\nu\beta)}(\frac{\lambda}{2}n) | PS \rangle &= H_{Tg}(x, \xi) \bar{U}(P'S') \frac{\bar{P}^{([\mu} i \Delta^{\alpha]} \sigma^{\nu\beta)}}{M} U(PS) \\
&+ \tilde{H}_{Tg}(x, \xi) \bar{U}(P'S') \frac{\bar{P}^{([\mu} \Delta^{\alpha]} \bar{P}^{[\nu} \Delta^{\beta])}}{M} \frac{1}{M^2} U(PS) \\
&+ E_{Tg}(x, \xi) \bar{U}(P'S') \frac{\bar{P}^{([\mu} \Delta^{\alpha]} \gamma^{[\nu} \Delta^{\beta])}}{M} \frac{1}{M} U(PS) \\
&+ \tilde{E}_{Tg}(x, \xi) \bar{U}(P'S') \frac{\bar{P}^{([\mu} \Delta^{\alpha]} \gamma^{[\nu} \bar{P}^{\beta])}}{M} \frac{1}{M} U(PS) + \dots \quad (5)
\end{aligned}$$

where the dependence of each distribution upon $t = \Delta^2$ and Q^2 is implicit. In the first equation, $[\mu\nu]$ denotes anti-symmetrization of the two indices and the ellipses represent higher twist structures. The quark helicity-flip distributions H_{Tq} , \tilde{H}_{Tq} , E_{Tq} and \tilde{E}_{Tq} can be selected by taking $\mu = +$ and $\nu = \perp$. The gauge link between the quark fields is not explicitly shown. Also by time-reversal symmetry and Hermiticity (complex conjugate), the quark distributions are real and even functions of ξ . In the second equation $[\mu\alpha]$ and $[\nu\beta]$ are antisymmetric pairs and (\dots) signifies symmetrization of the two and removal of the trace. Similarly, the gluon helicity-flip distributions H_{Tg} , \tilde{H}_{Tg} , E_{Tg} and \tilde{E}_{Tg} can be selected by taking $\mu = \nu = +$ and $\alpha, \beta = \perp$.

III. COUNTING OF INDEPENDENT FORM FACTORS OF QUARK OPERATORS

The quark and gluon operators defined above transform as irreducible representations of the Lorentz group. That is the reason for the (anti)symmetrization of the indices and removal of the traces. In this section we first briefly discuss the enumeration of independent elements of tensors with certain symmetry type and trace conditions. Then we consider the number of independent form factors for the matrix elements of the corresponding operators.

The number of independent elements of a tensor of rank n in n -dimensional (or d -dim) space is d^n . That of a totally symmetric tensor is $C_n^{d+n-1} = (d+n-1)!/(n!(d-1)!)$, and that of a totally antisymmetric tensor is $C_n^d = d!/(n!(d-n)!)$. The number of traceless conditions for a generic rank n tensor ($n \geq 2$) (in d -dim) is $C_2^n d^{n-2}$, while that for a rank r symmetric tensor is C_{n-2}^{d+n-3} . (This can be easily shown from recognizing that this number is the same as the number of independent elements of a rank $n-2$ symmetric tensor, since a

traceless condition is simply contracting two indices of the tensor.) Therefore, the number of independent elements of a rank- n symmetric traceless tensor in n -dim is $C_r^{d+n-1} - C_{n-2}^{d+n-3} = (n+1)^2$. In the notation of [9], where square parentheses are used to denote (the number of independent elements of) tensors with traces and usual (round) parentheses are used to denote (that of) traceless tensors, and the first number in a parenthesis is the number of symmetrized indices and the second number is that of anti-symmetrized indices, the above conclusion can be written as

$$(n, 0) = [n, 0] - [n-2, 0] .$$

The same result can be obtained by recognizing that such tensors furnish $\{\frac{n}{2}, \frac{n}{2}\}$ representations of the Lorentz group [10, 11], with their elements written as $T^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_n}$ (here we use curly brackets instead of usual parentheses to avoid confusion with the above notation). Such representations $\{A, B\}$ have $(2A+1)(2B+1)$ independent elements. As far as the J^{PC} properties and the number of independent matrix elements of the corresponding operators are concerned, they have one to one correspondence with Weyl spinors $\{A, B\}$ (e.g., [10, 11]).

For example, the symmetric quark operators in Eq. (2),

$$\mathcal{O}_q^{\mu_1 \dots \mu_n} = \bar{\psi}(0) i \overleftrightarrow{\mathcal{D}}^{(\mu_1} \dots i \overleftrightarrow{\mathcal{D}}^{\mu_n)} \gamma^{\mu_n} \psi(0) , \quad (6)$$

form totally symmetric tensor representations of the Lorentz group. Each $\mathcal{O}_{qT}^{\mu_1 \dots \mu_n}$ is a tensor of $(n, 0)$, and furnishes a $\{n/2, n/2\}$ representation of Lorentz group and has $(n+1)^2$ independent components.[11]

On the other hand, the operators \mathcal{O}_{qT} in (3) are rank $d = n+1$ traceless tensors in 4-dim with symmetric indices μ_1, \dots, μ_n and are antisymmetric in μ_n, α . For such operators

$$\mathcal{O}_{qT}^{\mu_1 \dots \mu_{n-1} \mu_n \alpha} = \bar{\psi}(0) i \overleftrightarrow{\mathcal{D}}^{(\mu_1} \dots i \overleftrightarrow{\mathcal{D}}^{\mu_{n-1}} \sigma^{\mu_n) \alpha} \psi(0), \quad (7)$$

the number of independent elements, labelled $(n, 1)$, can be worked out by the tensor method as shown in Appendix A. The result is $(n, 1) = 2 \times n(n+2)$. This is the same as the number of independent elements of the representation $\{\frac{n-1}{2}, \frac{n+1}{2}\}$ (with element $T^{\alpha_1 \dots \alpha_{n-1}}_{\beta_1 \dots \beta_{n+1}}$) and (plus) $\{\frac{n+1}{2}, \frac{n-1}{2}\}$ (with element $T^{\alpha_1 \dots \alpha_{n+1}}_{\beta_1 \dots \beta_{n-1}}$).

The J^{PC} content of the operators (2) and the numeration of the resulting independent form factors were discussed in [6]. Following the same method, one can analyze the operators in Eq. (7).

First one goes to the crossed channel where the operator serves as a source for creating a particle-antiparticle pair (matrix elements $\langle P\bar{P} | \mathcal{O}^{\mu_1 \dots \mu_k \mu \alpha} | 0 \rangle$, where $k+1 \equiv n$). The allowed J^{PC} values in this case are enumerated as the following (similar to the discussion in [6]): For $J^{PC}(L)$ values of $P\bar{P}$, $P = (-1)^{L+1}$, $C = (-1)^{L+S}$, and $S = 0, 1$. Thus in terms of J , when $S = 0$: $L = J$, $P = (-1)^J + 1$, $C = (-1)^J, (-1)^{J+1}$, and when $S = 1$: $L = J \pm 1$, $P = (-1)^J$, $C = (-1)^J$.

The J^{PC} of the operators (7) can be classified as the following: The representation $\{A, B\}$ has angular momentum $J = |A - B|, |A - B| + 1, \dots, A + B$ (since $\vec{J} = \vec{A} + \vec{B}$). The natural parity of the operator is $P = (-1)^J$, while the charge conjugation C of γ^μ , (each) $i \overleftrightarrow{\mathcal{D}}^\mu$, and $\sigma^{\mu\nu}$ are all -1 . Since parity transforms $A \leftrightarrow B$, for each J , both \pm parities are allowed. Then the representations $\{\frac{n-1}{2}, \frac{n+1}{2}\}$ and $\{\frac{n+1}{2}, \frac{n-1}{2}\}$ have $J = 1, 2, \dots, n$, $P = \pm$ and $C = (-1)^n$. The J^{PC} content of the operators and the $J^{PC}(L)$ values of the cross channel $P\bar{P}$ system are

n	$\mathcal{O}^{\mu_1 \dots \mu_n \nu}$	$P\bar{P}(J^{PC}(L))$
1	$1^{+-}, 1^{--}$	$1^{++}(1), 1^{+-}(1), 1^{--}(0), 1^{--}(2)$
2	$1^{++}, 1^{-+}, 2^{++}, 2^{-+}$	$2^{++}(1), 2^{++}(3), 2^{-+}(2), 2^{-+}(2)$
3	$1^{+-}, 1^{--}, 2^{+-}, 2^{--}, 3^{+-}, 3^{--}$	$3^{++}(3), 3^{+-}(3), 3^{--}(2), 3^{--}(4)$
4	$1^{++}, 1^{-+}, 2^{++}, 2^{-+}, 3^{++}, 3^{-+}, 4^{++}, 4^{-+}$	$4^{++}(3), 4^{++}(5), 4^{-+}(4), 4^{-+}(4)$
\vdots	\vdots	\vdots
n	$1^{+(-)^n}, 1^{-(-)^n}, \dots, n^{+(-)^n}, n^{-(-)^n}$	$n^{(-)^n(-)^n}(n-1), n^{(-)^n(-)^n}(n+1),$ $n^{(-)^{n+1}(-)^n}(n), n^{(-)^{n+1}(-)^{n+1}}(n)$
i.e.		
$n = \text{odd}$	$1^{+-}, 1^{--}, \dots, n^{+-}, n^{--}$	$n^{--}(n\pm 1), n^{++}(n), n^{+-}(n)$
$n = \text{even}$	$1^{++}, 1^{-+}, \dots, n^{++}, n^{-+}$	$n^{++}(n\pm 1), n^{-+}(n), n^{--}(n)$

For each J^{PC} , the number of independent form factors in matrix elements $\langle P' | \mathcal{O}^{\mu_1 \dots \mu_n \alpha} | P \rangle$ is determined by the number of independent amplitudes for the creation process (in the cross channel). For example, for $n = 1$, both 1^{+-} and 1^{--} sources are effective. While 1^{+-} can only create one state, the 1^{--} source can create two independent states (with $L = 0$ and 2). Therefore, there are three independent matrix elements. For $n = 2$, sources 1^{++} , 2^{++} , and 2^{-+} are effective. Both 1^{++} and 2^{-+} sources can only create one state, while the 2^{++} source can again produce two states (with $L = 1$ and 3). So there are four independent matrix elements. A list of matrix elements in terms of these quantum numbers can be generated as

$$\begin{aligned}
n = 1, & J^{PC}(L) = 1^{+-}(1), 1^{--}(0), 1^{--}(2) \\
n = 2, & J^{PC}(L) = 1^{++}(1); 2^{++}(1), 2^{++}(3), 2^{-+}(2) \\
n = 3, & J^{PC}(L) = 1^{+-}(1), 1^{--}(0), 1^{--}(2); 2^{--}(2); 3^{+-}(3), 3^{--}(2), 3^{--}(4) \\
n = 4, & J^{PC}(L) = 1^{++}(1); 2^{++}(1), 2^{++}(3), 2^{-+}(2); 3^{++}(3); 4^{++}(3), 4^{++}(5), 4^{-+}(4) \\
& \dots
\end{aligned}$$

This pattern can be extended and we have the enumeration of the independent form factors in $\langle P' | \mathcal{O}^{\mu_1 \dots \mu_k \mu \nu} | P \rangle$ as, where the $(\times 2)$ represents the two different $L (= J \pm 1)$ values for each J ,

n	Matched J^{PC}	Total Number
1	$1^{+-}, 1^{--}(\times 2)$	$(1 + 2) = 3$
2	$1^{++}; 2^{++}(\times 2), 2^{-+}$	$(1) + (1 + 2) = 4$
3	$1^{+-}, 1^{--}(\times 2); 2^{--}; 3^{+-}, 3^{--}(\times 2)$	$(1 + 2) + (1) + (1 + 2) = 7$
4	$1^{++}; 2^{++}(\times 2), 2^{-+}; 3^{++}; 4^{++}(\times 2), 4^{-+}$	$(1) + (1 + 2) + (1) + (1 + 2) = 8$
\dots	\dots	\dots
$n = \text{odd}$	$1^{+-}, 1^{--}(\times 2); 2^{--}; \dots; n^{+-}, n^{--}(\times 2)$	$(1 + 2) + (1) + \dots + (1 + 2)$ $= \frac{n+1}{2} \times 3 + \frac{n-1}{2} = 2n + 1$
$n = \text{even}$	$1^{++}; 2^{++}(\times 2), 2^{-+}; \dots; n^{++}(\times 2), n^{-+}$	$(1) + (1 + 2) + \dots + (1 + 2)$ $= \frac{n}{2} \times 3 + \frac{n}{2} = 2n$

Time reversal invariance does not impose any constraint in the crossed channel counting. However, when the same result is applied to the direct channel, the number reflects that after applying the time reversal symmetry. (Please also see discussions in Section IV and Appendix B.)

IV. FORM FACTORS OF TWIST-TWO HELICITY-FLIP QUARK OPERATORS

As was indicated in [3, 6], time reversal invariance might (or might not) introduce new constraints on the form factors thus further limit their numbers. In appendix B we look at the effect it has on the helicity-flip operators, and explicitly showed that it does not pose further limits on the matrix elements of the lowest rank operator in Eq. (7), and there are only four types of terms in the expansion of the matrix elements (see for example, Eq. (B13)). The higher rank operators in (7) (with additional covariant derivatives) will have factors of \overline{P} and Δ after taking matrix elements between p' and p , coming from the covariant derivatives[2, 12]. From Hermiticity requirements, after taking the hermitian/time reversal, each factor of Δ^μ will introduce a factor of -1 while p^μ factor will not. Therefore, the overall sign factor resulting from the time reversal/Hermitian operation that comes from the covariant derivatives (plus the extra factor of Δ from the anti-symmetric part in the C_3 and C_4 terms), in the matrix elements, will be $(-1)^l$ with l the number of factors of Δ present.

For the matrix element $\langle P' | \mathcal{O}^{\alpha\mu_1\mu_2\cdots\mu_n} | P \rangle$, (we have made a rearrangement of the indices that result only in a possible overall negative sign), just like the discussion leading to (B6), there are still only four types of terms in the expansion of the matrix element because of their Lorentz/tensorial structure, namely terms of the format $\overline{U}[\gamma^\alpha, \gamma^{\mu_1}]U \sim \sigma^{\mu\nu}$ (similar to the C_1 term in (B13)), $\overline{U}[\gamma^\alpha, \overline{P}^{\mu_1}]U \sim \gamma^\mu$ (similar to the C_2 term in (B13)), $\overline{U}[\gamma^\alpha, \Delta^{\mu_1}]U \sim \gamma^\mu$ (similar to the C_3 term in (B13)), and $\overline{U}[\overline{P}^\alpha, \Delta^{\mu_1}]U \sim \overline{U}U$ (similar to the C_4 term in (B13)). The \sim sign means having the same properties under time reversal (hermitian, complex conjugate). The operator $\mathcal{O}^{\alpha\mu_1\mu_2\cdots\mu_n}$ is odd (“−”) overall under time reversal because of the anti-symmetric part $\sigma_{\mu\nu}$ (the covariant derivatives are even). Same is the first type of terms, while the rest three are even (“+”). Therefore to give the matrix elements the proper signs under time reversal/Hermitian, namely overall odd (“−”), only certain numbers of factors of Δ would appear thus limiting the numbers of form factors and the structures associated with them. We thus have the following table:

TABLE I: Enumeration from time-reversal/Hermiticity considerations

Term	\hat{T}	Sign allowed from Factors of \bar{P} and Δ			
		$n=1$	\dots	$n = 2k + 1$	$n = 2k$
$\bar{U}[\gamma^\alpha, \gamma^{\mu_1}]U$	$-$	$+$		$(-1)^l$, with $l = 0, 2, \dots, 2k(= n-1)$	$(-1)^l$, with $l = 0, 2, \dots, 2k-2(= n-2)$
# allowed		1		$k + 1 = \frac{n+1}{2}$	$k = \frac{n}{2}$
$\bar{U}[\gamma^\alpha, \bar{P}^{\mu_1}]U$	$+$	$+$		$(-1)^l$, with $l = 1, 3, \dots, 2k-1(= n-2)$	$(-1)^l$, with $l = 1, 3, \dots, 2k-1(= n-1)$
# allowed		0		$k = \frac{n-1}{2}$	$k = \frac{n}{2}$
$\bar{U}[\gamma^\alpha, \Delta^{\mu_1}]U$	$+$	$-$		$(-1)^l$, with $l = 0, 2, \dots, 2k(= n-1)$	$(-1)^l$, with $l = 0, 2, \dots, 2k-2(= n-2)$
# allowed		1		$k + 1 = \frac{n+1}{2}$	$k = \frac{n}{2}$
$\bar{U}[\bar{P}^\alpha, \Delta^{\mu_1}]U$	$+$	$-$		$(-1)^l$, with $l = 0, 2, \dots, 2k(= n-1)$	$(-1)^l$, with $l = 0, 2, \dots, 2k-2(= n-2)$
# allowed		1		$k + 1 = \frac{n+1}{2}$	$k = \frac{n}{2}$
Total #		3		$4k+3 = 2n+1$	$4k = 2n$

The above is completely consistent and thus confirms our counting of the numbers of independent form factors of the quark operators in section III. Further, it is now clear that the matrix elements of these operators must have its expansion into kinematic factors, Lorentz/tensorial (symmetry) structure factors and independent form factors in the form of

$$\begin{aligned}
 \langle P' | \mathcal{O}^{\alpha\mu_1\mu_2\cdots\mu_n} | P \rangle = & \bar{U}(P') \sigma^{\alpha(\mu_1} U(P) \sum_{i=0}^{[\frac{n+1}{2}]} A_{n,2i-1} \Delta^{\mu_2} \Delta^{\mu_3} \dots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \dots \bar{P}^{\mu_n}) \\
 & + \bar{U}(P') [\gamma^\alpha, \bar{P}^{(\mu_1}] U(P) \sum_{i=0}^{[\frac{n}{2}]} B_{n,2i} \Delta^{\mu_2} \Delta^{\mu_3} \dots \Delta^{\mu_{2i}} \bar{P}^{\mu_{2i+1}} \dots \bar{P}^{\mu_n}) \\
 & + \bar{U}(P') [\gamma^\alpha, \Delta^{(\mu_1}] U(P) \sum_{i=0}^{[\frac{n+1}{2}]} i C_{n,2i-1} \Delta^{\mu_2} \Delta^{\mu_3} \dots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \dots \bar{P}^{\mu_n}) \\
 & + \bar{U}(P') [\bar{P}^\alpha, \Delta^{(\mu_1}] U(P) \sum_{i=0}^{[\frac{n+1}{2}]} i D_{n,2i-1} \Delta^{\mu_2} \Delta^{\mu_3} \dots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \dots \bar{P}^{\mu_n}) .
 \end{aligned} \tag{8}$$

One can recover Eq. (4) easily by multiplying the above by $n^{\mu_1} n^{\mu_2} \dots n^{\mu_{n-1}}$ and converting the moments into the light-cone fraction space.

V. FORM FACTORS OF TWIST-TWO HELICITY-FLIP GLUON OPERATORS

The tensor operators that flips gluon spin are

$$\mathcal{O}_{gT}^{\mu\alpha\nu\beta\mu_1\cdots\mu_n} = F^{(\mu\alpha} i \overleftrightarrow{\mathcal{D}}^{\mu_1} \cdots i \overleftrightarrow{\mathcal{D}}^{\mu_n} F^{\nu)\beta}, \quad (9)$$

where μ, ν and μ_1, \dots, μ_n are symmetrized while μ and α as well as ν and β are anti-symmetric pairs, and the operator is also rendered traceless (note we relabelled \mathcal{O}_{gT} in (3) using $\mu_1 \rightarrow \mu$, $\mu_n \rightarrow \nu$, and μ_2, \dots, μ_{n-1} to μ_1, \dots, μ_n). Equation (9) corresponds to the Young tableau $(n+2, 2)$, as well as the Weyl representations $\{\frac{n}{2}, \frac{n+4}{2}\}$ (with element labelled $T^{\alpha_1 \cdots \alpha_n}_{\dot{\beta}_1 \cdots \dot{\beta}_{n+4}}$) and $\{\frac{n+4}{2}, \frac{n}{2}\}$ (with elements labelled $T^{\alpha_1 \cdots \alpha_{n+4}}_{\dot{\beta}_1 \cdots \dot{\beta}_n}$). That is,

$$\begin{array}{|c|c|c|c|c|} \hline \mu & \nu & \mu_1 & \cdots & \mu_n \\ \hline \alpha & \beta & & & \\ \hline \end{array} - \text{trace term} = (n+2, 2) .$$

According to the standard group theoretical result, $[n+2, 2] = (n+1)(n+4)(n+5)$. As shown in Appendix A, the number of trace conditions for the tensor $[n+2, 2]$ is $(n+1)(n+2)(n+5)$, resulting in the number of independent elements of the traceless tensor $(n+2, 2)$ being $2(n+1)(n+5)$. This is verified by the enumeration of the elements of the Weyl representations $\{\frac{n}{2}, \frac{n+4}{2}\}$ and $\{\frac{n+4}{2}, \frac{n}{2}\}$, each of which being $(n+1)(n+5)$.

Now let us discuss the J^{PC} content of these gluon helicity-flip operators. The angular momentum J obviously can take on values of $2, 3, \dots, n+2$. Similar to the discussion in section III, both parity values are allowed. The charge conjugation of the gluon field bilinear FF is even ("++") (while $F^{\mu\nu}$ transforms just like $\sigma^{\mu\nu}$ under time reversal [13, 14]). Therefore comparing with the allowed J^{PC} values in the cross channel, just as was done in section III, we have the following enumeration for the number of independent form factors,

n	$\mathcal{O}^{\mu\alpha\nu\beta\mu_1\cdots\mu_n}$	Matched ($J^{PC}(L)$)	Enumeration
0	$2^{++}, 2^{-+}$	$2^{++}(1), 2^{++}(3), 2^{-+}(2)$	3
1	$2^{+-}, 2^{--}, 3^{+-}, 3^{--}$	$2^{--}(2), 3^{+-}(3), 3^{--}(2), 3^{--}(4)$	$1+3=4$
2	$2^{++}, 2^{-+}, 3^{++}, 3^{-+}$ $4^{++}, 4^{-+}$	$2^{++}(1), 2^{++}(3), 2^{-+}(2), 3^{++}(3)$ $4^{++}(3), 4^{++}(5), 4^{-+}(4)$	$3+1+3=7$
3	$2^{+-}, 2^{--}, 3^{+-}, 3^{--}$ $4^{+-}, 4^{--}, 5^{+-}, 5^{--}$	$2^{--}(2), 3^{+-}(3), 3^{--}(2), 3^{--}(4)$ $4^{--}(4), 5^{+-}(5), 5^{--}(4), 5^{--}(6)$	$1+3+1+3=8$
...
$n = \text{odd}$	$2^{+-}, 2^{--}, \dots,$ $[n+2]^{+-}, [n+2]^{--}$	$2^{--}(2), 3^{+-}(3), 3^{--} \times 2, \dots,$ $[n+2]^{+-}(n+2), [n+2]^{--} \times 2$	$(1+3) \times \frac{n+1}{2}$ $= 2(n+1)$
$n = \text{even}$	$2^{++}, 2^{-+}, \dots,$ $[n+2]^{++}, [n+2]^{-+}$	$2^{++} \times 2, 2^{-+}(2), \dots,$ $[n+2]^{++} \times 2, [n+2]^{-+}(n+2)$	$3 + (1+3) \times \frac{n}{2}$ $= 2n+3$

We notice that the number of form factors for the gluon operators is the same as that for the quark operators with corresponding spin.

It is obvious from comparison of Eq. 4 and Eq. 5 that the only difference between the kinematic and tensorial factors of the quark and gluon matrix elements is a factor of

$\bar{P}i\Delta$. More detailed analysis confirms this, and following the similar discussions in sections III,IV and appendix B, we find that the form factor expansion of the matrix elements of the twist-two helicity-flip gluon operators is

$$\begin{aligned}
\langle P' | \mathcal{O}_{gT}^{\mu\alpha\nu\beta\mu_1\ldots\mu_n} | P \rangle = & \bar{U}(P') \sigma^{(\mu\alpha} U(P) \sum_{i=0}^{[\frac{n}{2}]} i A_{n,2i-1} \Delta^{\mu_1} \Delta^{\mu_2} \ldots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \ldots \bar{P}^{\mu_n} [\bar{P}^{\nu)}, \Delta^\beta] \\
& + \bar{U}(P') [\gamma^{(\mu}, \bar{P}^\alpha] U(P) \sum_{i=0}^{[\frac{n-1}{2}]} i B_{n,2i} \Delta^{\mu_1} \Delta^{\mu_2} \ldots \Delta^{\mu_{2i}} \bar{P}^{\mu_{2i+1}} \ldots \bar{P}^{\mu_n} [\bar{P}^{\nu)}, \Delta^\beta] \\
& + \bar{U}(P') [\gamma^{(\mu}, \Delta^\alpha] U(P) \sum_{i=0}^{[\frac{n}{2}]} C_{n,2i-1} \Delta^{\mu_1} \Delta^{\mu_2} \ldots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \ldots \bar{P}^{\mu_n} [\bar{P}^{\nu)}, \Delta^\beta] \\
& + \bar{U}(P') [\bar{P}^{(\mu}, \Delta^\alpha] U(P) \sum_{i=0}^{[\frac{n}{2}]} D_{n,2i-1} \Delta^{\mu_1} \Delta^{\mu_2} \ldots \Delta^{\mu_{2i-1}} \bar{P}^{\mu_{2i}} \ldots \bar{P}^{\mu_n} [\bar{P}^{\nu)}, \Delta^\beta] .
\end{aligned} \tag{10}$$

We have used the same symbols as for the quark form factors.

VI. SUMMARY AND OUTLOOK

By matching of quantum numbers between cross channels, we count the number of independent form factors in the off-forward matrix elements of the quark and gluon helicity-flip operators ((7) and (9)). We found that for a rank- n ($n \geq 2$) quark helicity-flip operator that number is $2n - 1$ if n is even and $2n - 2$ if n is odd. For the rank- n ($n \geq 4$) gluon helicity-flip operator that number is $2n - 5$ and $2n - 6$, respectively. Also, the matrix elements of these operators have their expansion into these form factors, together with the appropriate tensor structure and kinematic factors, in the form of equations (8) and (10). The independent form factors emerging this way can be related to the moments of the GPDs and thus can be used to define and extract the GPDs themselves. One can further pursue this route to cast more light on the intrinsic structure of these operators and/or form factors, eg, power counting (of light cone wave-functions) [15].

As a final note, after this work was completed, a paper dealing with similar topics emerged by P. Hagler [16], in which similar results for the quark operators were obtained and were consistent with ours.

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APPENDIX A: NUMBER OF INDEPENDENT ELEMENTS OF TENSORS

Using Young tableau, one can calculate the number of independent elements of $[n, 1]$ [17] in 4-D, and the matching representation enumerations. Following the discussions in [9], the systematic decomposition of tensor $[n, 1]$ into traceless tensors is as follows (omitting the comma between numbers):

$$\begin{aligned}
 [11] &= (11) \\
 &\quad 6 \quad 6 \\
 [21] &= (21) + (10) \\
 &\quad 20 \quad 16 \quad 4 \\
 [31] &= (31) + (11) + (20) \\
 &\quad 45 \quad 30 \quad 6 \quad 9 \\
 [41] &= (41) + (21) + (30) + (10) \\
 &\quad 84 \quad 48 \quad 16 \quad 16 \quad 4 \\
 [51] &= (51) + (31) + (11) + (40) + (20) \\
 &\quad 140 \quad 70 \quad 30 \quad 6 \quad 25 \quad 9 \\
 &\dots
 \end{aligned}$$

$$\begin{aligned}
 n \text{ odd} \quad [n, 1] &= (n, 1) + (n-2, 1) + \dots + (11) + (n-1, 0) + (n-3, 0) + \dots + (20) \\
 n \text{ even} \quad [n, 1] &= (n, 1) + (n-2, 1) + \dots + (21) + (n-1, 0) + (n-3, 0) + \dots + (10)
 \end{aligned}$$

To prove the result that $(n, 1) = 2n(n+2)$, one can use the iteration method. From

$$[n, 1] = (n, 1) + (n-2, 1) + \dots + (n-1, 0) + (n-3, 0) + \dots ,$$

one has

$$\begin{aligned}
 [n+2, 1] &= (n+2, 1) + (n, 1) + \dots + (n+1, 0) + (n-1, 0) + \dots \\
 &= (n+2, 1) + (n+1, 0) + [n, 1] .
 \end{aligned}$$

Let $D(n) = 2n(n+2)$, then it is apparent from explicit calculation/enumeration that $D(n) = (n, 1)$ is valid for $n = 1, 2, 3$. Given $[n, 2] = \frac{n}{2}(n+2)(n+3)$ and $(n, 0) = (n+1)^2$, one has

$$\begin{aligned}
 [n+2, 1] - [n, 1] - (n+1, 0) &= \frac{n+2}{2}(n+4)(n+5) - \frac{n}{2}(n+2)(n+3) - (n+2)^2 \\
 &= 2(n+2)(n+4) = D(n+2) .
 \end{aligned}$$

The Proof of the iteration of

$$[n, 1] = (n, 1) + [n-2, 1] + (n-1, 0) \tag{A1}$$

is as the following:

The trace conditions for a Young tableau is the regular removal of two boxes in the graphic representation/calculation of Young tableaux(eg, see [9, 18]). And for $[n, 1]$, which is represented as

$$\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_n \\ \hline \alpha & & & \end{array} = \overline{\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_n \\ \hline \alpha & & & \end{array}} - \begin{array}{|c|c|c|c|} \hline \alpha & \mu_2 & \cdots & \mu_n \\ \hline \mu_1 & & & \end{array} ,$$

where the double line means explicitly symmetric, there are only two ways for the contraction.

No.1. Contracting any pair out of $\mu_2, \mu_3, \dots, \mu_n$. Because of the explicit symmetry of $\mu_2, \mu_3, \dots, \mu_n$, each contraction is the same as contracting μ_{n-1} and μ_n , This results in

$$\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-2} \\ \hline \alpha & & & \\ \hline \end{array} = [n-2, 1] .$$

No.2. Contracting one of 1 or α with any one of $\mu_2, \mu_3, \dots, \mu_n$. Because of the anti-symmetry of μ_1 and α , we have

$$\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdots & \mu_n \\ \hline \alpha & & & \\ \hline \end{array} = - \begin{array}{|c|c|c|c|} \hline \alpha & \cdot & \mu_3 & \cdots & \mu_n \\ \hline \cdot & & & & \\ \hline \end{array} = - \begin{array}{|c|c|c|c|} \hline \alpha & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array} .$$

Thus each and all of the contraction is equivalent to contracting α and μ_n , which results in a mixed symmetry of $\mu_2, \mu_3, \dots, \mu_{n-1}$ (with μ_1 still in front, and this is not $[n-1, 0]$). Explicitly, it is,

$$\begin{aligned} \begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array} &= \overline{\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array}} - \overline{\begin{array}{|c|c|c|c|} \hline \cdot & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \mu_1 & & & & \\ \hline \end{array}} \\ &= \overline{\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array}} - \frac{1}{2} \left[\overline{\begin{array}{|c|c|c|c|} \hline \cdot & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \mu_1 & & & & \\ \hline \end{array}} + \overline{\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array}} \right] \\ &\quad + \frac{1}{2} \left[\overline{\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \cdot & & & & \\ \hline \end{array}} - \overline{\begin{array}{|c|c|c|c|} \hline \cdot & \mu_2 & \cdots & \mu_{n-1} & \cdot \\ \hline \mu_1 & & & & \\ \hline \end{array}} \right] , \end{aligned}$$

where we have rewritten the second term into a symmetric combination of μ_1 and (the contracted) α (the first square parentheses) and an anti-symmetric one (the second square parentheses).

The first square parentheses represents a pair of contraction in the now all symmetrized $\mu_1, \mu_2, \dots, \mu_{n-1}$, and together with the first term, they are actually

$$\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_{n-1} \\ \hline \end{array} - \text{trace terms} = (n-1, 0) .$$

On the other hand, the second square parentheses term is symmetric in μ_2, \dots, μ_{n-1} but anti-symmetric in $\mu_1 \leftrightarrow \mu_2$ Therefore it is indeed

$$\begin{array}{|c|c|c|c|} \hline \mu_2 & \mu_3 & \cdots & \mu_{n-1} \\ \hline \mu_1 & & & \\ \hline \end{array} = [n-2, 1] .$$

This of course is the same as No.1, ie, they are the same trace(less) conditions.

Combine No.1 and No.2 we find that the number of trace(less) conditions for $[n, 1]$ is $[n-2, 1] + (n-1, 0)$, thus proving (A1).

Therefore, we have the following table as the resulting enumerations: (Also included are the numbers of trace(less) conditions for each tensor, denoted by Tr , and the column *Representation* is the enumeration of the corresponding Lorentz group representation(s) $\{A, B\}$ (and $\{B, A\}$.)

TABLE II: The enumeration of independent elements of tensors with n -symmetrized and one pair of anti-symmetrized indices.

n	Generic	Tr	Traceless	Representation
1	N/A	0	$(1, 1) = 6$	$\{0, 1\} + \{1, 0\} = 3 \times 2$
2	$[2, 1] = 20$	4	$(2, 1) = 16$	$\{\frac{1}{2}, \frac{3}{2}\} + \{\frac{3}{2}, \frac{1}{2}\} = 4 \times 2$
3	$[3, 1] = 45$	15	$(3, 1) = 30$	$\{1, 2\} + \{2, 1\} = 15 \times 2$
4	$[4, 1] = 84$	36	$(4, 1) = 48$	$\{\frac{3}{2}, \frac{5}{2}\} + \{\frac{5}{2}, \frac{3}{2}\} = 24 \times 2$
5	$[5, 1] = 140$	70	$(5, 1) = 70$	$\{2, 3\} + \{3, 2\} = 35 \times 2$
...
n	$[n, 1] = \frac{n}{2}(n+2)(n+3)$	$\frac{n}{2}(n-1)(n+2)$ $= C_2^n(n+2)$	$(n, 1) = 2n(n+2)$	$\{\frac{n-1}{2}, \frac{n+1}{2}\} + \{\frac{n+1}{2}, \frac{n-1}{2}\}$ $= n(n+2) \times 2$

Similar to the case earlier, following the same discussions in [9], the systematic decomposition of tensor $[n+2, 2]$ into traceless tensors is as follows:

$$\begin{aligned}
[22] &= (22) + (20) + (00) \\
&\quad 20 \quad 10 \quad 9 \quad 1 \\
[32] &= (32) + (21) + (30) + (10) \\
&\quad 60 \quad 24 \quad 16 \quad 16 \quad 4 \\
[42] &= (42) + (40) + (20) + (31) + (22) + (20) + (00) \\
&\quad 126 \quad 42 \quad 25 \quad 9 \quad 30 \quad \underbrace{10 \quad 9 \quad 1}_{[22]} \\
[52] &= (52) + (50) + (30) + (41) + (32) + (21) + (30) + (10) \\
&\quad 224 \quad 64 \quad 36 \quad 16 \quad 48 \quad \underbrace{24 \quad 16 \quad 16 \quad 4}_{[32]} \\
[62] &= (62) + (60) + (40) + (51) + (42) + (40) + (20) + (31) + (22) + (20) + (00) \\
&\quad 360 \quad 90 \quad 49 \quad 25 \quad 70 \quad \underbrace{42 \quad 25 \quad 9 \quad 30 \quad 10 \quad 9 \quad 1}_{[42]} \\
&\quad \dots \\
[n+2, 2] &= (n+2, 2) + (n+2, 0) + (n, 0) + (n+1, 1) + [n, 2].
\end{aligned}$$

The proof of the iteration of $[n+2, 2]$ would be similar to before.

Again by calculating the number of elements in Young tableau, we have $[n+2, 2] = (n+1)(n+4)(n+5)$, $[n, 2] = (n-1)(n+2)(n+3)$, $(n+2, 0) = (n+3)^2$, $(n, 0) = (n+1)^2$, and $(n+1, 1) = 2(n+1)(n+3)$.

APPENDIX B: DISCUSSIONS ON TIME REVERSAL

Let \hat{T} be the time reversal operator in Hilbert space. Then its operation on p^μ will result in $p^0 \rightarrow p^0$ while $\vec{p} \rightarrow -\vec{p}$, that is, $\hat{T}p^\mu \rightarrow p_\mu$. For spin wave functions (of Dirac spinors) $\psi(t, \vec{x})$ one needs $\hat{T}\psi(t, \vec{x})\hat{T}^{-1} \leftrightarrow T\psi(t, \vec{x}) \rightarrow \psi(-t, \vec{x})$, where T is a 4×4 time reversal matrix acting on spinors (in Dirac space). T is a so-called *anti-linear* or *anti-unitary* operator with $T^\dagger = T^{-1}$ but $T(c - \text{number}) = (c - \text{number})^*T$. It reverses the momentum of a particle as well as its spin, eg, for the Fermion annihilation operators $Tb_{\vec{p},\lambda}T^{-1} = b_{-\vec{p},-\lambda}$ and $Td_{\vec{p},\lambda}T^{-1} = d_{-\vec{p},-\lambda}$.

Labelling $\tilde{p} = (p^0, -\vec{p})$, one must have $T|P\rangle = \eta_T|\tilde{p}\rangle \equiv e^{i\phi}|\tilde{p}\rangle$, where η_T is a pure phase factor. Because \hat{T} commutes with Lorentz boost (a change in \vec{p}), η_T does not depend on p and is a fixed phase. With $|p'\rangle = e^{i\chi}|P\rangle$, and the fact that T acts on c -numbers is equivalent to taking the complex conjugate, one has $T|p'\rangle = e^{-i\chi}e^{i\phi}|\tilde{p}\rangle = e^{-2i\chi}e^{i\phi}|\tilde{p}'\rangle$. Therefore, one can always choose $\chi = \phi/2$ and thus

$$T|p^{(\prime)}\rangle = |\tilde{p}^{(\prime)}\rangle .$$

Explicitly,

$$\psi(t, \vec{x}) = \sum_{\lambda} \int \frac{d^3\vec{k}}{2k^0(2\pi)^3} \left(e^{-ik \cdot x} b_{k,\lambda} U(k, \lambda) + e^{ik \cdot x} d_{k,\lambda}^\dagger V(k, \lambda) \right)$$

and

$$\hat{T}\psi(t, \vec{x})\hat{T}^{-1} = \sum_{\lambda} \int \frac{d^3\vec{k}}{2k^0(2\pi)^3} \left(e^{+ik \cdot x} b_{\tilde{k},\lambda} U^*(k, \lambda) + e^{-ik \cdot x} d_{\tilde{k},\lambda}^\dagger V^*(k, \lambda) \right) .$$

Thus one needs a time reversal matrix T such that [13] $U^*(k, \lambda) = TU(\tilde{k}, \lambda)$, that is, $TU(k) = U^*(\tilde{k})$. And one has

$$\begin{aligned} \hat{T}\psi(t, \vec{x})\hat{T}^{-1} &= T \sum_{\lambda} \int \frac{d^3\vec{k}}{2k^0(2\pi)^3} \left(e^{+ik \cdot x} b_{\tilde{k},\lambda} U(\tilde{k}, \lambda) + e^{-ik \cdot x} d_{\tilde{k},\lambda}^\dagger V(\tilde{k}, \lambda) \right) \\ &= T \sum_{\lambda} \int \frac{d^3\vec{k}}{2k^0(2\pi)^3} \left(e^{+i\tilde{k} \cdot x} b_{\tilde{k},\lambda} U(\tilde{k}, \lambda) + e^{-i\tilde{k} \cdot x} d_{\tilde{k},\lambda}^\dagger V(\tilde{k}, \lambda) \right) , \end{aligned}$$

where in the last step we have changed the integration variable from \vec{k} to $-\vec{k}$. Relabel $\tilde{k} \cdot x = k^0 x^0 + \vec{k} \cdot \vec{x} = -k \cdot \tilde{x}$, and again change the integration variable $\vec{k} \rightarrow -\vec{k}$, we have

$$\hat{T}\psi(t, \vec{x})\hat{T}^{-1} = T \sum_{\lambda} \int \frac{d^3\vec{k}}{2k^0(2\pi)^3} \left(e^{-i\tilde{k} \cdot \tilde{x}} b_{\tilde{k},\lambda} U(\tilde{k}, \lambda) + e^{i\tilde{k} \cdot \tilde{x}} d_{\tilde{k},\lambda}^\dagger V(\tilde{k}, \lambda) \right) .$$

That is

$$\hat{T}\psi(t, \vec{x})\hat{T}^{-1} = T\psi(\tilde{x}) = T\psi(-t, \vec{x}) . \quad (\text{B1})$$

At the same time, since \hat{T} acts in Hilbert space it commutes with γ^0 , we have

$$\begin{aligned} \hat{T}\bar{\psi}(x)\hat{T}^{-1} &= \hat{T}\psi^\dagger\gamma^0\hat{T}^{-1} = \hat{T}\psi^\dagger\hat{T}^{-1}\gamma^0 \\ &= (\hat{T}\psi\hat{T}^{-1})^\dagger\gamma^0 = (T\psi(\tilde{x}))^\dagger\gamma^0 = \psi^\dagger(\tilde{x})T^\dagger\gamma^0 \\ &= \bar{\psi}(\tilde{x})T^\dagger . \end{aligned} \quad (\text{B2})$$

As an example, take $\mathcal{O}^\mu = \bar{\psi}\gamma^\mu\psi$, we have

$$\begin{aligned}\hat{T}\mathcal{O}^\mu\hat{T}^{-1} &= \hat{T}\bar{\psi}(x)\hat{T}^{-1}\hat{T}\gamma^\mu\hat{T}^{-1}\hat{T}\psi(x)\hat{T}^{-1} \\ &= \bar{\psi}(\tilde{x})T^\dagger(\gamma^\mu)^*T\psi(\tilde{x}).\end{aligned}$$

In Dirac representation, $T = i\gamma^1\gamma^3$. Therefore under $T^\dagger(\gamma^\mu)^*T$, $\gamma^0 \rightarrow \gamma^0$ because γ^0 is real and commutes with T , $\gamma^2 \rightarrow -\gamma^2$ because γ^2 is imaginary and commutes with T , and $\gamma^{1/3} \rightarrow -\gamma^{1/3}$ because $\gamma^{1/3}$ is real and anti-commutes with T . Thus, since $(\gamma^i)^2 = I$ ($i = 1, 2, 3$),

$$\hat{T}\bar{\psi}\gamma^\mu\psi\hat{T}^{-1} = \bar{\psi}\gamma_\mu\psi. \quad (\text{B3})$$

Similarly, for $\mathcal{O}^{\mu\nu} = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$,

$$\hat{T}\bar{\psi}\sigma^{\mu\nu}\psi\hat{T}^{-1} = -\bar{\psi}\sigma_{\mu\nu}\psi. \quad (\text{B4})$$

And also

$$\begin{aligned}U^*(\hat{T}p) &= TU(P) \quad (\text{or } U(\hat{T}p) = T^*U^*(p)) \\ \bar{U}^*(\hat{T}p') &= \bar{U}(P')T^\dagger\end{aligned} \quad (\text{B5})$$

For completeness, the covariant derivative $i\overleftrightarrow{\mathcal{D}}$ is even ("+") under time reversal.

As an explicit example, let us now discuss the (additional) constraints time reversal invariance might have on the matrix elements of the lowest rank operator in (7) $\mathcal{O}^{\mu\nu} = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$. To assure the antisymmetry, only terms of the following types according to their Lorentz/tensorial structure will appear

$$\begin{aligned}\langle P'|\mathcal{O}^{\mu\nu}|P\rangle &= c_1\bar{U}(P')\sigma^{\mu\nu}U(P) + c_2\bar{U}(P')[\gamma^\mu, \bar{P}^\nu]U(P) \\ &+ c_3\bar{U}(P')[\gamma^\mu, \Delta^\nu]U(P) + c_4\bar{U}(P')[\bar{P}^\mu, \Delta^\nu]U(P),\end{aligned} \quad (\text{B6})$$

where we have used the notation

$$[\gamma^\mu, \bar{P}^\nu] = \gamma^\mu\bar{P}^\nu - \gamma^\nu\bar{P}^\mu$$

and similar for $[\gamma^\mu, \Delta^\nu]$ and $[\bar{P}^\mu, \Delta^\nu]$.

The fact that $\mathcal{O}^{\mu\nu}$ is a Hermitian operator means

$$\langle P'|\mathcal{O}^{\mu\nu}|P\rangle^\dagger = \langle p|(\mathcal{O}^{\mu\nu})^\dagger|p'\rangle = \langle p|\mathcal{O}^{\mu\nu}|p'\rangle. \quad (\text{B7})$$

Because \bar{P} is symmetric in p and p' , and Δ is anti-symmetric in them, the signs of c_1 and c_2 terms would remain the same under Hermitian operation, while those of the other two would change. That is, we have

$$\begin{aligned}\langle P'|\mathcal{O}^{\mu\nu}|P\rangle^\dagger &= \langle p|\mathcal{O}^{\mu\nu}|p'\rangle \\ &= c_1\bar{U}(P)\sigma^{\mu\nu}U(P') + c_2\bar{U}(P)[\gamma^\mu, \bar{P}^\nu]U(P') \\ &- c_3\bar{U}(P)[\gamma^\mu, \Delta^\nu]U(P') - c_4\bar{U}(P)[\bar{P}^\mu, \Delta^\nu]U(P').\end{aligned} \quad (\text{B8})$$

For the c_1 term, direct application of the Hermitian would give,

$$\begin{aligned}\langle P'|\mathcal{O}^{\mu\nu}|P\rangle^\dagger &= \dots + c_1^*[\bar{U}(P')\sigma^{\mu\nu}U(P)]^\dagger + \dots = \dots + c_1^*U(P)^\dagger(\sigma^{\mu\nu})^\dagger\bar{U}(P')^\dagger + \dots \\ &= \dots + c_1^*U(P)^\dagger\gamma^0\sigma^{\mu\nu}\gamma^0\gamma^0U(P') + \dots = \dots + c_1^*\bar{U}(P)\sigma^{\mu\nu}U(P') + \dots; (\text{B9})\end{aligned}$$

Therefore one has $c_1^* = c_1$ which means c_1 is real, and let us label $c_1 \equiv C_1$. Similarly, for the c_2 term, direct Hermitian gives

$$\begin{aligned}\langle P' | \mathcal{O}^{\mu\nu} | P \rangle^\dagger &= \dots + c_2^* U(P)^\dagger (\gamma^\mu \bar{P}^\nu - \gamma^\nu \bar{P}^\mu)^\dagger \bar{U}(P')^\dagger + \dots \\ &= \dots + c_2^* U(P)^\dagger (\gamma^0 \gamma^\mu \gamma^0 \bar{P}^\nu - \gamma^0 \gamma^\nu \gamma^0 \bar{P}^\mu) \gamma^0 U(P') + \dots \\ &= \dots + c_2^* \bar{U}(P) [\gamma^\mu, \bar{P}^\nu] U(P') .\end{aligned}\tag{B10}$$

Therefore one has $c_2^* = c_2$ which means c_2 is also real, and let us label $c_2 \equiv C_2$. For the c_3 term, on the other hand, direct Hermitian gives

$$\begin{aligned}\langle P' | \mathcal{O}^{\mu\nu} | P \rangle^\dagger &= \dots + c_3^* U(P)^\dagger [\gamma^\mu, \Delta^\nu]^\dagger \bar{U}(P')^\dagger + \dots \\ &= \dots + c_3^* U(P)^\dagger \gamma^0 [\gamma^\mu, \Delta^\nu] \gamma^0 U(P') + \dots \\ &= \dots + c_3^* \bar{U}(P) [\gamma^\mu, \Delta^\nu] U(P') .\end{aligned}\tag{B11}$$

Compared with (B8) one has $c_3^* = -c_3$ which means c_3 is pure imaginary, and one defines $c_3 \equiv iC_3$ where C_3 is real.

And for the c_4 term,

$$\begin{aligned}\langle P' | \mathcal{O}^{\mu\nu} | P \rangle^\dagger &= \dots + c_4^* U(P)^\dagger [\bar{P}^\mu, \Delta^\nu]^\dagger \bar{U}(P')^\dagger + \dots \\ &= \dots + c_4^* U(P)^\dagger \gamma^0 [\bar{P}^\mu, \Delta^\nu] \gamma^0 U(P') + \dots \\ &= \dots + c_4^* \bar{U}(P) [\bar{P}^\mu, \Delta^\nu] U(P') .\end{aligned}\tag{B12}$$

Therefore one has $c_4^* = -c_4$ which means c_4 is also pure imaginary, and one defines $c_4 \equiv iC_4$ where C_4 is real.

Thus, the matrix element actually has the form

$$\begin{aligned}\langle P' | \mathcal{O}^{\mu\nu} | P \rangle &= C_1 \bar{U}(P') \sigma^{\mu\nu} U(P) + C_2 \bar{U}(P') [\gamma^\mu, \bar{P}^\nu] U(P) \\ &\quad + iC_3 \bar{U}(P') [\gamma^\mu, \Delta^\nu] U(P) + iC_4 \bar{U}(P') [\bar{P}^\mu, \Delta^\nu] U(P) ,\end{aligned}\tag{B13}$$

where all the $C_i (i = 1, 2, 3, 4)$ are real.

On the other hand, the requirement of time reversal invariance means

$$\langle P' | \mathcal{O}^{\mu\nu} | P \rangle = \langle P' | T^{-1} T \mathcal{O}^{\mu\nu} T^{-1} T | P \rangle^* = \langle TP' | T \mathcal{O}^{\mu\nu} T^{-1} | TP \rangle^*$$

The C_1 term in $\langle TP' | T \mathcal{O}^{\mu\nu} T^{-1} | TP \rangle^*$ is (see (B4))

$$\begin{aligned}-(C_1^* \bar{U}(TP') \sigma_{\mu\nu} U(TP))^* &= -C_1 \bar{U}^*(TP') \sigma_{\mu\nu}^* U^*(TP) \\ &= -C_1 \bar{U}(P') T^\dagger \sigma_{\mu\nu}^* T U(P) ,\end{aligned}\tag{B14}$$

where we have used (B5) in the last step. Since $T^\dagger = T = i\gamma^1 \gamma^3$, we have, similar to (B3),

$$\begin{aligned}T^\dagger \sigma_{\mu\nu}^* T &= i\gamma^1 \gamma^3 \left(-\frac{i}{2}\right) [\gamma_\mu, \gamma_\nu]^* i\gamma^1 \gamma^3 \\ &= \frac{i}{2} (\gamma^1 \gamma^3 \gamma_\mu^* \gamma_\nu^* \gamma^1 \gamma^3 - \gamma^1 \gamma^3 \gamma_\nu^* \gamma_\mu^* \gamma^1 \gamma^3) \\ &= -\sigma^{\mu\nu} .\end{aligned}\tag{B15}$$

It is clear that time reversal invariance does not impose further constraints on this term, while the structure $\sigma^{\mu\nu}$ is odd ("") under time reversal.

Because the time reversal takes C -numbers to its complex conjugate, together with (B3), we have under time reversal $[\gamma^\mu, \bar{P}^\nu] \rightarrow [\gamma_\mu, \bar{P}^\nu]$, and the C_2 term becomes

$$\begin{aligned}\langle TP' | TC_2 [\gamma^\mu, \bar{P}^\nu] T^{-1} | TP \rangle^* &= (C_2^* \bar{U}(TP') [\gamma_\mu, \bar{P}^\nu] U(TP))^* \\ &= C_2 \bar{U}^*(TP') [\gamma_\mu, \bar{P}^\nu]^* U^*(TP) \\ &= C_2 \bar{U}(P') T^\dagger (\gamma_\mu^* \bar{P}^\nu - \gamma_\nu^* \bar{P}^\mu) TU(P) .\end{aligned}\quad (\text{B16})$$

Similarly, we would have

$$T^\dagger (\gamma_\mu^* \bar{P}^\nu - \gamma_\nu^* \bar{P}^\mu) T = [\gamma^\mu, \bar{P}^\nu]$$

Thus we need $C_2 = -C_2$ and hence we must have $C_2 \equiv 0$.

The C_3 term, on the other hand, is

$$\begin{aligned}\langle TP' | TiC_3 [\gamma^\mu, \Delta^\nu] T^{-1} | TP \rangle^* &= (-iC_3^* \bar{U}(TP') [\gamma_\mu, \Delta^\nu] U(TP))^* \\ &= iC_3 \bar{U}^*(TP') [\gamma_\mu, \Delta^\nu]^* U^*(TP) \\ &= iC_3 \bar{U}(P') T^\dagger (\gamma_\mu^* \Delta^\nu - \gamma_\nu^* \Delta^\mu) TU(P) ,\end{aligned}\quad (\text{B17})$$

with

$$T^\dagger (\gamma_\mu^* \Delta^\nu - \gamma_\nu^* \Delta^\mu) T = (\gamma^\mu \Delta^\nu - \gamma^\nu \Delta^\mu) = [\gamma^\mu, \Delta^\nu] .$$

It is clear then that time reversal invariance does not impose further constraints on this term.

Similarly, the C_4 term is

$$\begin{aligned}\langle TP' | TiC_4 [\bar{P}^\mu, \Delta^\nu] T^{-1} | TP \rangle^* &= (-iC_4^* \bar{U}(TP') [\bar{P}^\mu, \Delta^\nu] U(TP))^* \\ &= iC_4 \bar{U}^*(TP') [\bar{P}^\mu, \Delta^\nu]^* U^*(TP) \\ &= iC_4 \bar{U}(P') T^\dagger [\bar{P}^\mu, \Delta^\nu] TU(P) ,\end{aligned}\quad (\text{B18})$$

and obviously

$$T^\dagger [\bar{P}^\mu, \Delta^\nu] T = [\bar{P}^\mu, \Delta^\nu] .$$

This means time reversal invariance does not impose further constraints on this term either.

Therefore we confirmed there are three independent form factors for the operator $\mathcal{O}_{qT}^{\mu\nu}$, consistent with the general counting in section III. Similar discussions can be carried out for other helicity-flip operators, and from the combination of Hermiticity and time reversal symmetry, the general counting result will all stand.

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